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NON-STATIONARY OPTIMALITY CONDITIONS IN STRUCTURAL DESIGN†

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Abstract—In some cases, optimal structural design corresponds to non-analytical points for which the first variation of volume or cost does not necessarily vanish. Such cases may occur, for instance, when optimal design is aimed at maximizing the lowest eigenvalue (natural frequency, buckling load) and a multimodal solution occurs, or when the aim is to minimize the maximum stress intensity. The present paper discusses such cases and provides the relevant optimality conditions.

1. INTRODUCTION

In a recent paper[1], Olhoff and Rasmussen have discussed the significance of multiple eigenvalues in connection with the optimal design of structural members with eigenvalue constraints. They have shown, for example, that the optimal design of unconstrained clamped columns occurs when the smallest two buckling loads coalesce and that, as a result, the classical solution of Tadjbaksh and Keller[2] based on a single eigenvalue is invalid. Corresponding observations have subsequently been made by Szelag and Mróz[3], who considered optimal design of vibrating beams with unspecified *a priori* support action and who found that both analytical extremum points and multimodal points may occur for optimal solutions.

The present note is intended to throw additional light on the nature of the results of [1] and [3] and to derive optimality conditions on the basis of a more general, and perhaps more instructive approach. In addition, the design for local stress constraint is discussed and sufficient optimality conditions are derived. These conditions are a typical feature of min–max or max–min problems, and their investigation is therefore an indispensable feature in the solution of such problems.

The existence of non-stationary optima, away from the boundary of an admissible domain, can easily be demonstrated by a simple example. Let the numbers x_i ($i = 1,2, \ldots n$) be subject to the constraint $\Sigma x_i = 1$, and find the set $x_i = x_{0i}$ such that $(x_{0i})_{\max}$ becomes as small as possible. The solution $x_{0i} = 1/n$ ($i = 1,2, \ldots n$) is obvious, as is the proof of optimality, which, however, does not involve stationarity. As an illustration, a mass particle may be attached to two mutually perpendicular spring systems with spring constants k_1 and k_2, respectively. Then, for prescribed total "cost" $k = k_1 + k_2$ the fundamental frequency in the plane of the springs is maximized if $k_1 = k_2 = k/2$, and for the corresponding problem in three dimensions the optimal solution is given by $k_1 = k_2 = k_3 = k/3$; in both cases the solution involves multiple eigenvalues[12].

The situation is similar if Πx_i is prescribed. For example, an unbraced prismatic column of rectangular cross section, whose total area is prescribed, is obviously strongest if that cross section is square (assuming suitable boundary conditions). In fact, engineers have long accepted the general, though not infallible, principle that optimal design involves simultaneous failure into several modes.

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The concept may readily be extended to continua. Let $y(x)$ be an integrable function, whose supremum is to be minimized on the segment $(0, l)$ subject to the constraint $\int_0^l y(x)\,dx = 1$; this leads once again to the non-stationary solution $y = y_0 = 1/l$. In fact, considering any function $y(x) = 1/l + c\eta(x)$, with $\int_0^l \eta\,dx = 0$, and denoting sub $\eta(x) = \alpha > 0$, inf $\eta(x) = -\beta < 0$, then $I(y) \equiv \sup y(x) = 1/l + c\alpha > 1/l$ for $c > 0$ and $I(y) = 1/l - c\beta > 1/l$ for $c > 0$, and hence $dI/d|c| > 0$ as $c \to 0$ irrespective of whether the limit is approached from above or from below. An example of this type of non-stationary optimality is presented in Section 4.

2. OPTIMAL DESIGN FOR MULTIPLE EIGENVALUES IN CONSERVATIVE SYSTEMS

Consider the problem of optimal design of a beam or plate with specified in-plane forces, loading, and boundary conditions. The design variable $\alpha(x)$ represents a cross-sectional property such as a varying beam width or height, or the sheet thickness of a sandwich plate. Consider the typical problem of maximizing the eigenvalue λ_i, which is governed by the system of differential equations and boundary conditions

$$\mathbf{L}\mathbf{u}_i - \lambda_i \mathbf{G}\mathbf{u}_i = 0 \qquad (i = 1,2,\ldots) \tag{1}$$

and subject to the condition of prescribed upper bound on volume of cost

$$C \equiv \int_\tau A(\alpha)\,dx \leqq C_0. \tag{2}$$

In eqn (1) $\mathbf{L}(\alpha)$ and $\mathbf{G}(\alpha)$ are linear, self-adjoint, positive definite operators in the eigenfunction $\mathbf{u}_i(x)$, and algebraic functions of the design variable α, while the eigenfunctions $\mathbf{u}_i$ are normalized in the sense of

$$(\mathbf{u}_i, \mathbf{G}\mathbf{u}_i) = 1. \tag{3}$$

We also note that the self-adjointness of the operators implies the existence of scalar functions U and H, quadratic in $\mathbf{u}$, such that

$$\begin{aligned}(\mathbf{u}, \mathbf{L}\mathbf{u}) &= \int_\tau U(\mathbf{u})\,dx\\ (\mathbf{u}, \mathbf{G}\mathbf{u}) &= \int_\tau H(\mathbf{u})\,dx\end{aligned} \tag{4}$$

in which the integration extends over the region τ of the body.

Consider now a "neighboring" design $\alpha + \dot{\alpha}$ (associated with $\mathbf{u}_i + \dot{\mathbf{u}}_i$ and $\lambda_i + \dot{\lambda}_i$); then, by varying eqn (1), we obtain

$$\mathbf{L}\dot{\mathbf{u}}_i - \lambda_i \mathbf{G}\dot{\mathbf{u}}_i = -[\dot{\alpha}\mathbf{L}_\alpha]\mathbf{u}_i + \lambda_i[\dot{\alpha}G_\alpha]\mathbf{u}_i + \dot{\lambda}_i \mathbf{G}\mathbf{u}_i \tag{5}$$

in which dots denote first variations and a subscript α denotes differentiation with respect to α. An explicit expression for $\dot{\lambda}_i$ is obtained by comparing eqns (1) and (5) after taking the inner product with $\dot{\mathbf{u}}_i$ and $\mathbf{u}_i$, respectively. In view of eqn (3) we obtain

$$\begin{aligned}\dot{\lambda}_i &= (\mathbf{u}_i, [\dot{\alpha}\mathbf{L}_\alpha]\mathbf{u}_i) - \lambda_i(\mathbf{u}_i, [\dot{\alpha}\mathbf{G}_\alpha]\mathbf{u}_i)\\ &= \int_\tau [U_\alpha(\mathbf{u}_i) - \lambda_i H_\alpha(\mathbf{u}_i)]\dot{\alpha}\,dx\end{aligned} \tag{6}$$

in which we have again used the self-adjointness of the operators $\mathbf{L}$ and $\mathbf{G}$ and their derivatives with respect to α.

The technically most significant problem is to maximize the lowest eigenvalue λ_1. In this case there are two possibilities:

(1) A maximum occurs when the lowest eigenvalue $\lambda_1 > 0$ is single and the associated

eigenfunction $\mathbf{u}_1(x)$ is unique (except for the sign). Prior to the publication of [1] all authors appear to have concerned themselves primarily with this case, at least by implication.

(2) The lowest eigenvalue represents a multiple—say, double— root, to which correspond two linearly independent eigenmodes $\mathbf{u}_1$ and $\mathbf{u}_2$. This second case, which has received very little attention in the literature, is the main subject of this section.

Optimality criteria for the first case were derived, e.g. in Refs. [1, 2], in which the stationarity condition

$$\dot{\lambda}_1 = 0 \tag{7}$$

for all $\dot{\alpha}(x)$ satisfying

$$\int_\tau A_\alpha \dot{\alpha}\, \mathrm{d}x = 0 \tag{8}$$

was shown to take the form

$$P_1 \equiv U_\alpha(\mathbf{u}_1) - \lambda_1 H_\alpha(\mathbf{u}_1) - \beta_1{}^2 A_\alpha = 0, \qquad (x \in \tau). \tag{9}$$

The conditions for global maximum of λ_1 are analogous to those derived in [4], where the minimum cost design was considered for specified first frequency of free vibrations. It was shown that if **L** and **G** are negative definite with respect to α, then the structure attains its minimum cost at the stationary point, for which (9) is satisfied. Associated with this global optimum is the local maximality condition (for constant volume)

$$\begin{aligned}\ddot{\lambda}_1 = &\int_\tau [U_{\alpha\alpha}(\mathbf{u}_1) - \lambda_1 H_{\alpha\alpha}(\mathbf{u}_1) - \beta_1{}^2 A_{\alpha\alpha}]\dot{\alpha}^2\, \mathrm{d}x \\ &- 2\int_\tau [U(\dot{\mathbf{u}}_1) - \lambda_1 H(\dot{\mathbf{u}}_1)]\, \mathrm{d}x < 0,\end{aligned} \tag{10}$$

for which a sufficient condition is

$$U_{\alpha\alpha}(\mathbf{u}_1) - \lambda_1 H_{\alpha\alpha}(\mathbf{u}_1) - \beta_1{}^2 A_{\alpha\alpha} < 0, \qquad (x \in \tau) \tag{11}$$

since the second integral in (10) is positive in view of Rayleigh's principle and of the fact that $\dot{u}_1$ is a kinematically admissible field. The sufficiency condition (11) for the local maximum of λ_1 is in accord with the results of [4].

The condition of optimality for the second case of coincident eigenvalues λ_1 and λ_2 is less restrictive. Although it is of course still possible for both λ_1 and λ_2 to be stationary (see eqns 9 and 11), a local optimum may now also exist without either eigenvalue being stationary. In fact we may assume that for some variation $\dot{\alpha}(x)$ neither $\dot{\lambda}_1$ nor $\dot{\lambda}_2$ vanishes; then it is sufficient to postulate further that $\dot{\lambda}_1$ and $\dot{\lambda}_2$ be of opposite sign. In that case the neighboring design $\alpha + \dot{\alpha}$ is associated with two distinct eigenvalues $\lambda_1 + \dot{\lambda}_1$ and $\lambda_2 + \dot{\lambda}_2$, the smaller of which is less than $\lambda_1 = \lambda_2$, and a local non-stationary optimum is therefore reached (see Fig. 1).

Let the point of coincidence be identified by

$$\lambda_1 = \lambda_2 = \lambda \tag{12}$$

and let it be associated with the two independent modes $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively. (Note: the subscripts do not necessarily refer to the ordering of the eigenvalues by their magnitudes near the point of coincidence.) Then, according to the discussion above, a local optimum is reached if

$$\dot{\lambda}_1 \dot{\lambda}_2 \leqq 0 \tag{13}$$

for any $\dot{\alpha}(x)$ satisfying eqn (8). Equivalent to inequality (13) (for $\dot{\lambda}_1 \neq 0$), but easier to handle is

$$\frac{\dot{\lambda}_2}{\dot{\lambda}_1} = \frac{\int_\tau P_2 \dot{\alpha}\, \mathrm{d}x}{\int_\tau P_1 \dot{\alpha}\, \mathrm{d}x} \leqq 0 \tag{14}$$

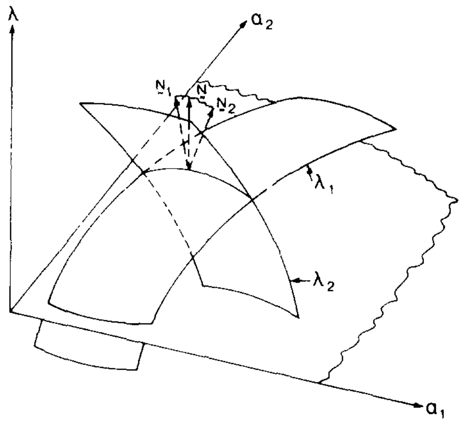

Fig. 1. Intersecting surfaces.

in which P_1 and P_2 are defined in eqn (9) (the latter by exchanging subscripts). If we let $P_1(x) \equiv P(x)$ and define $f(x)$ by

$$P_2(x) = f(x)\,P(x), \tag{15}$$

then (14) becomes

$$\frac{\int_\tau fP\dot{\alpha}\,\mathrm{d}x}{\int_\tau P\dot{\alpha}\,\mathrm{d}x} \leqq 0, \tag{16}$$

which must be satisfied for all variations $\dot{\alpha}(x)$.

The establishment of the necessary and sufficient conditions to satisfy (16) follows standard arguments of the calculus of variations. In fact, let

$$\dot{\alpha}(x) = \delta(x - x_1), \tag{17}$$

in which δ represents the Dirac function; then, for $P(x_1) \neq 0$, (16) implies

$$f_1 \equiv f(x_1) \leqq 0. \tag{18}$$

Since x_1 is arbitrary this means that f must be nonpositive throughout the region.

Similarly, for

$$\dot{\alpha}(x) = \alpha_1\delta(x - x_1) + \alpha_2\delta(x - x_2) \tag{19}$$

the condition (16) assumes the form

$$f_1\frac{\alpha_1 P(x_1) + r\alpha_2 P(x_2)}{\alpha_1 P(x_1) + \alpha_2 P(x_2)} \leqq 0 \qquad r = f_2/f_1 \tag{20}$$

for any value of α_1 and α_2. Moreover, if we define, for $P(x_1) \neq 0$,

$$g(x) = \frac{\alpha_2 P(x_2)}{\alpha_1 P(x_1)} \tag{21}$$

and invoke (18), then (20) is reduced to

$$\frac{1+rg}{1+g} \geqq 0, \qquad (g \in -\infty, \infty). \tag{22}$$

It is easy to show that for any value of $r \neq 1$ it is always possible to find a value of g such that (22) is violated. It therefore follows that $r = 1$ is a necessary condition for (14) to be satisfied, or equivalently,

$$f(x) = \frac{P_2(x)}{P_1(x)} = -\omega^2, \qquad (x \in \tau). \tag{23}$$

The sufficiency of eqn (23) is readily verified by substitution in eqn (16).†

It is interesting to note, as a perhaps not surprising by-product of eqn (23), that the ratio of $\dot{\lambda}_2$ to $\dot{\lambda}_1$ is the same for all (not trivially vanishing) variations $\dot{\alpha}$ which do not follow the "edge" between the "surfaces" identifying $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ (see Fig. 1). If, on the other hand, both $\dot{\lambda}_1 = \dot{\lambda}_2 = 0$ for some variation $\dot{\alpha}$, then this variation is tangent to the edge identified by $\lambda_1(\alpha) \equiv \lambda_2(\alpha)$, and in that case eqn (10), with eqn (11) as a sufficient condition, must hold for both $\ddot{\lambda}_1$ and $\ddot{\lambda}_2$.

After rearrangements of the constants eqn (23) can be put into the standard form

$$(1-\gamma)[U_\alpha(\mathbf{u}_1) - \lambda H_\alpha(\mathbf{u}_1)] + \gamma[U_\alpha(\mathbf{u}_2) - \lambda H_\alpha(\mathbf{u}_2) = \beta^2 A_\alpha \qquad (x \in \tau,\ \gamma \in 0, 1) \tag{24}$$

which corresponds to eqn (13) in [1] for the unconstrained case. The problem treated in [1] concerns the buckling of a column of prescribed shape (say, circular), but not size of the cross section. In nondimensional form the relevant operators and quadratic forms are

$$\begin{aligned} &\mathbf{L}\mathbf{u}_i = (\alpha^2 u_i'')'' \quad (x \in 0, 1); \qquad u_i = u_i' = 0 \quad (x = 0, 1) \\ &\mathbf{G}\mathbf{u}_i = -u_i'' \quad (x \in 0, 1) \quad A(\alpha) = \alpha \\ &U(\mathbf{u}_i) = \alpha^2 u_i''^2 \qquad H(u_i) = u_i'^2 \end{aligned} \tag{25}$$

(with primes denoting derivatives with respect to x), and the optimality condition eqn (24) takes the simple form

$$2\alpha(1-\gamma)u_1''^2 + 2\alpha\gamma u_2''^2 = \beta^2 \qquad (x \in 0, 1;\ \gamma \in 0, 1). \tag{26}$$

This is identical with the result of [1]. The latter was derived through a variational process covering the stationary case of all variations $\dot{\alpha}$ which follow the intersecting "curve" $\lambda_1 \equiv \lambda_2$. The additional restriction $\gamma \epsilon (0, 1)$ (see eqn (24)) completes the coverage by extending it to the non-stationary case in which $\dot{\alpha}$ contains a component orthogonal to the intersection.

3. FLUTTER

The developments of the previous section can readily be extended to nonconservative problems, in which multiple modes, with nonstationary properties, may also occur in addition to stationary single modes. As an example we select the case of the optimal design of a structure under non-conservative loading conditions—say, a column subjected to a follower force—whose stationary analysis was carried out by Vepa[5] and by Odeh and Tadjbakhsh[6], while Claudon[7] has recently performed a numerical investigation of the case of a double root.

It is well known, as a result of the work of Ziegler *et al.*[8], that the stability of a structure under these conditions must be analyzed on a dynamic basis. Letting the response function $\mathbf{U}(x, t)$ be of the form

$$\mathbf{U}(x, t) = \mathbf{u}(x)\, e^{i\omega t} \tag{27}$$

we reduce the problem to the solution of

$$\mathbf{L}\mathbf{u} - \lambda \mathbf{G}\mathbf{u} \equiv \omega^2 \mathbf{M}\mathbf{u} = \mathbf{0} \tag{28}$$

†Equation (23) can also be shown to be directly derivable by means of a lemma on p. 13 of [13]. The authors are indebted to one of the referees for bringing this lemma to their attention.

in which, for the example selected, L and M are positive definite self adjoint linear operators in u. Equivalent to eqn (28) is

$$\mathbf{Lv} - \lambda \mathbf{G}^T\mathbf{v} - \omega^2\mathbf{Mv} = \mathbf{0}, \tag{29}$$

where $\mathbf{G}^T$ is adjoint to G. Without loss of generality it is convenient to normalize u and v in the sense that $(\mathbf{v}, \mathbf{Gu}) = (\mathbf{u}, \mathbf{G}^T\mathbf{v}) = 1$. We also note, for future reference, that

$$(\mathbf{u}, \mathbf{Lv}) = (\mathbf{v}, \mathbf{Lu}) = \int_\tau U(\mathbf{u}, \mathbf{v})\, dx$$
$$(\mathbf{u}, \mathbf{Mv}) = (\mathbf{v}, \mathbf{Mu}) = \int_\tau K(\mathbf{u}, \mathbf{v})\, dx \tag{30}$$

in which U and K are bilinear in u and v and may be identified with the strain energy and kinetic energy densities, respectively. If $\mathbf{G} = \mathbf{G}^T$ then ω^2 is real for all values of λ. The structure is stable when $\omega_1{}^2 > 0$, unstable when $\omega_1{}^2 < 0$, and the critical ("divergence") state is reached when $\omega_1 = 0$. This case is readily seen to be covered by the discussion of the previous section.

If $\mathbf{G} \neq \mathbf{G}^T$, that is, if the problem is nonconservative, the structure is stable if the smallest two roots $(\omega_1')^2$ and $(\omega_1'')^2$ are real, distinct, and positive. In the nonconservative case, however, instability may also occur when $(\omega_1')^2$ and $(\omega_1'')^2$ become complex conjugate. The critical state ("flutter") corresponds to the double root $(\omega_1')^2 = (\omega_1'')^2 = \omega_1{}^2 > 0$.

The analytical condition governing the onset of flutter can be established by considering u and λ_1 to be functions of ω^2. By differentiating eqn (28) with respect to ω^2, with asterisks denoting derivatives, we obtain

$$\mathbf{Lu}^* - \lambda \mathbf{Gu}^* - \omega^2\mathbf{Mu}^* = \lambda^*\mathbf{Gu} + \mathbf{Mu}. \tag{31}$$

At the point at which ω_1' and ω_1'' coalesce, λ^* satisfies

$$\lambda^* = 0 \tag{32}$$

since this corresponds to the largest value of λ for which two real, though coincident, roots $(\omega_1')^2$ and $(\omega_1'')^2$ exist. Thus a comparison of eqns (29) and (31), by applying the inner product with $\mathbf{u}^*$ to the former and with v to the latter, and consideration of eqns (32) and (30), lead to the flutter condition

$$\int_\tau K(\mathbf{u}, \mathbf{v})\, dx = 0, \tag{33}$$

which was previously established by Plaut[9] for a more restricted class of problems. The onset of dynamic instability is therefore governed by eqns (28) and (29) and the flutter condition eqn (33).

For optimization let L and M be functions of α. In analogy with the previous section we now obtain, by varying eqn (28) with respect to α,

$$\mathbf{L\dot{u}} - \lambda \mathbf{G\dot{u}} - \omega^2\mathbf{M\dot{u}} = -[\dot{\alpha}L_\alpha]\mathbf{u} + \dot{\lambda}\mathbf{Gu} + (\dot{\omega^2})\mathbf{Mu} + \omega^2[\dot{\alpha}\mathbf{M}_\alpha]\mathbf{u}, \tag{34}$$

and if we take the inner product of this equation with v, and of eqn (29) with $\dot{\mathbf{u}}$, and by invoking eqn (33), we establish

$$\dot{\lambda} = \int_\tau [U_\alpha(\mathbf{u}, \mathbf{v}) - \omega^2 K_\alpha(\mathbf{u}, \mathbf{v})]\dot{\alpha}\, dx. \tag{35}$$

An optimum is reached once again when eqn (7) is satisfied for all variations $\dot{\alpha}$ satisfying eqn (8); this leads to the condition of stationarity

$$U_\alpha(\mathbf{u}, \mathbf{v}) - \omega^2 K_\alpha(\mathbf{u}, \mathbf{v}) = \beta^2 A_\alpha \qquad (x \in \tau) \tag{36}$$

which was obtained, in more restrictive form, by Vepa[5], who investigated the problem of the optimal design of a cantilever column subject to a follower force.

Claudon[7] has noted that the solution given in [5] does not represent an optimum for the problem considered. Instead, the optimum condition occurs when the same eigenvalue λ is reached for two separate double roots ω_1^2 and ω_2^2, as any change in the design corresponds to an increase in one of the eigenvalues and a decrease in the other. His results are predominantly numerical.

Analytically this corresponds once again to satisfying eqns (12) and (13), in which λ_1 and λ_2 are the values of λ associated with the double roots ω_1^2 and ω_2^2, respectively. Application of a process entirely analogous to that of the previous section then leads to the optimality condition

$$(1-\gamma)[U_\alpha(\mathbf{u}_1, \mathbf{v}_1) - \omega_1^2 K_\alpha(\mathbf{u}_1, \mathbf{v}_1)] + \gamma[U_\alpha(\mathbf{u}_2, \mathbf{v}_2) - \omega_2^2 K_\alpha(\mathbf{u}_2, \mathbf{v}_2)] = \beta^2 A_\alpha$$

$$(x \in \tau;\ \gamma \in 0, 1) \qquad (37)$$

which corresponds to eqn (24) for the case of divergence. For the particular case treated, which is similar to the conservative case treated in [1], but with boundary conditions making the operator **G** non-selfadjoint, the optimality condition eqn (37) reduces to

$$(1-\gamma)(2\alpha u_1'' v_1'' - \omega_1^2 u_1 v_1) + \gamma(2\alpha u_2'' v_2'' - \omega_2^2 u_2 v_2) = \beta^2 \qquad (x \in 0, 1), \qquad (38)$$

from which eqn (26) can be recovered by setting $u_1 = v_1$, $u_2 = v_2$ and $\omega_1 = \omega_2 = 0$.

A comparison between the numerical results obtained by Claudon[7] and eqn (38) is inconclusive. The results clearly show that the stationarity condition eqn (36) is invalid. Some qualitative agreement with the nonstationary condition eqn (37) (or eqn 38) is discernible; nevertheless there is sufficient quantitative disagreement to raise some doubt as to the exact optimality of the results published in [7]. Since the computational technique employed in [7] is otherwise accurate and efficient it is possible that the discrepancy may be due to excessive sensitivity. Further studies along these lines are certainly in order and recommended.

4. DESIGN UNDER MAXIMUM STRESS INTENSITY CONSTRAINT

One of the most common problems in engineering is to determine the best design of a structure or of a machine element on the basis that, for given maximum stress intensity, its cost is to be reduced to a minimum, or, equivalently, that the maximum stress intensity is to be as small as possible for given cost. In this case the constraint is local, for given global objective function (or vice versa), whereas in the previous two sections both objective function and constraint are global. Nevertheless, it is shown in this section that for a broad class of problems it is possible to find an optimum; this optimum, however, is once again not stationary.

For this purpose it is useful to introduce the generalized stress **m**, which may represent a bending moment in the case of a beam, or a bending moment tensor in the case of a plate, or any other system of generalized internal stresses which are associated with a corresponding system of strains $\boldsymbol{\kappa}$. The total complementary energy E_c is given by

$$E_c = \int_\tau U_c(\mathbf{m}, \alpha)\, dx, \qquad (39)$$

in which U_c, the complementary energy density, is assumed to depend on the design parameter α, and the integral extends over the body as before. The stress **m** and strain $\boldsymbol{\kappa}$ are connected through the constitutive relations

$$\boldsymbol{\kappa} = \frac{\partial U_c}{\partial \mathbf{m}}. \qquad (40)$$

We also note, as pointed out in [10], that the stress intensity, Ω_c, that is, the average complementary energy density in the extreme fibers, is given by

$$\Omega_c = -\frac{\dfrac{\partial U_c}{\partial \alpha}}{\dfrac{\partial A}{\partial \alpha}} = -\frac{\partial U_c}{\partial A}. \tag{41}$$

If we now pose the preliminary problem of minimizing E_c for given volume or cost function, then, from eqns (39) and (8),

$$\dot{E}_c = \int_\tau \left(\frac{\partial U_c}{\partial \mathbf{m}}\right)^T \dot{\mathbf{m}} dx + \int_\tau \left(\frac{\partial U_c}{\partial \alpha} - \beta^2 \frac{\partial A}{\partial \alpha}\right) \dot{\alpha}\, dx = 0. \tag{42}$$

The first integral in this equation vanishes from the principle of virtual work because, by eqn (40), it represents the internal work of a compatible strain field relative to a self-equilibrated stress field. It therefore follows that eqn (42) is satisfied for arbitrary design variations $\dot{\alpha}$ if, and only if,

$$\Omega_c = \beta^2 \qquad (x \in \tau), \tag{43}$$

in which the definition of eqn (41) has been invoked. It is noted that eqn (43) represents the necessary condition for the complementary energy, or, equivalently, the total compliance, to become stationary.

In a great number of technically significant problems the strain energy density U_c is of the form

$$U_c(\mathbf{m}, A) = \frac{k}{A^n} f(\mathbf{m}) \qquad (n > 0). \tag{44}$$

For example, $n = 1$ may represent the case of a bar under tension or compression, of a sandwich beam or plate, or of a closed thin-walled section of given profile but varying thickness under torsion. Rectangular beams of given width and varying depth, or solid plates of varying depth, are represented by the case of $n = 3$.

By eqns (39), (41) and (44), E_c can then be expressed as

$$E_c = \frac{1}{n} \int_\tau \Omega_c A\, dx. \tag{45}$$

Let a design α satisfy eqn (43); then, by eqn (45), eqn (42) becomes

$$n\dot{E}_c = \int_\tau \dot{\Omega}_c A\, dx + + \int_\tau \Omega_c \dot{A}\, dx = 0. \tag{46}$$

The second integral in this equation vanishes by eqns (8) and (43), and the first integral therefore vanishes also. Since A is nowhere negative, and excluding the trivial case in which $\dot{\Omega}_c$ vanishes everywhere except at points of vanishing cross sections, it therefore follows that $\dot{\Omega}_c$ cannot everywhere be negative or zero, or, equivalently, that

$$(\dot{\Omega}_c)_{\max} > 0. \tag{47}$$

We can therefore draw the conclusion that for any other design $\alpha^* = \alpha + \dot{\alpha}$, $\Omega_c^* = \Omega + \dot{\Omega}_c \pm \ldots$, and within the linear approximation employed here,

$$(\Omega_c^*)_{\max} > \beta^2. \tag{48}$$

In other words, for any other design the safety factor is reduced. Equivalently, let an alternate design retain the same safety factor, that is, let

$$\Omega_c^* \leqq \beta^2. \tag{49}$$

This implies that $\dot{\Omega}_c$ cannot be positive, and, in view of eqn (43), and employing the same reasoning as before, it follows from eqn (46) that

$$\int \dot{A}\, dx > 0; \tag{50}$$

an alternate design employing the same safety factor must therefore cost more.

Consider finally an alternate design with the same (or larger) safety factor and the same cost, i.e. let eqns (8) and (49) both be satisfied. Then, from eqn (46), $\dot{E}_c < 0$, and eqn (42) is therefore violated. Since, in view of eqn (43), the second integral still vanishes, it follows that the first integral in eqn (42) must be negative.

We now assume that the external force system is identified by $\lambda \mathbf{F}_0$, in which $\mathbf{F}_0$ is a given force system and λ represents a common multiplier, which may be assumed, without loss of generality, to be positive. Let $\mathbf{u}$ represent the actual displacement field; then (42) must be replaced by

$$\dot{E}_c = \int_T \dot{\mathbf{m}}^T \boldsymbol{\kappa} dx = \dot{\lambda} \int_T \mathbf{F}_0{}^T \mathbf{u} dx < 0, \tag{51}$$

in which we have once again invoked the principle of virtual work since the strain field $\boldsymbol{\kappa}$ is compatible with the displacement field $\mathbf{u}$, and the stress field $\dot{\mathbf{m}}$ is in equilibrium with the force system $\dot{\lambda}\mathbf{F}_0$. The integral on the right side of eqn (51) is positive since, when multiplied by λ, it represents the total external work and is therefore equal to $2E_c$. It follows therefore that

$$\dot{\lambda} < 0, \tag{52}$$

which states that an alternate design of the same cost and safety factor carries a reduced load.

In any event, any interpretation leads to the conclusion that a design satisfying eqn (43), subject to the restriction of eqn (44), constitutes a local optimum in the sense that the situation becomes less favorable, within a linear approximation, if any change in the design is introduced. We are therefore once again dealing with a nonstationary condition of optimality; a similar conclusion, though on a restricted basis, was reached in [10].

5. CONCLUDING REMARKS

The equations developed in Sections 2 and 3, and notably optimality conditions of the type of eqns (24) or (26), have appeared in the literature before, and in connection with diverse problems. For example, Prager and Shield[11] have shown that under certain restrictive conditions equations of the type of eqn (26) may represent a sufficiency condition for optimality under dual constraints. It may also be noted that in the theory of perfect plasticity the strain rate vector is normal to the yield surface, and if that surface exhibits a corner then the strain rate vector lies in a plane region which is identified and bounded, as in eqn (24), by a constant γ between zero and unity.

The analogy between the results of Sections 2 and 3 and the theory of perfect plasticity is not as far-fetched as it may seem. Let eqn (24) be multiplied by $\dot{\alpha}(x)$ and integrated over the region; then, by eqns (6) and (8),

$$(1-\gamma)\dot{\lambda}_1 + \gamma\dot{\lambda}_2 = 0 \qquad (\gamma \in 0, 1) \tag{53}$$

which shows, once again, that $\dot{\lambda}_1$ and $\dot{\lambda}_2$ have opposite signs (unless they both vanish). Equation (7), or eqn (53), admit the following geometric interpretation:

Let N represent the normal to the "surface" identifying λ as a function of $\alpha(x)$, that is, let N be orthogonal to the "tangent plane" formed by a totality of all $\dot{\lambda}$. Then, for a single surface, the stationary condition $\dot{\lambda} = 0$ (eqn (7)) obviously implies that N is "vertical", that is, parallel to the λ axis in Fig. 1. On the other hand, if the optimal point lies on the intersection between two surfaces associated with λ_1 and λ_2, with normals N_1 and N_2 not both parallel to the λ axis, then there must exist a "vertical" normal **N** such that

$$\mathbf{N} = (1-\gamma)\mathbf{N}_1 + \gamma\mathbf{N}_2. \tag{54}$$

The plane identified by eqn (53) (not shown in Fig. 1) then is normal to **N** and hence "horizontal". The same discussion can easily be extended to intersections of higher order.

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